Properties of Peak Set Polynomials

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Abstract

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of $\{1, 2, \ldots, n\}$. An index i is a *peak* of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and let $P(\pi)$ be the set of peaks of π. Given any set S of positive integers we define $\mathcal{P}(S; n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}$. It has been shown that for all fixed subsets of positive integers S and all sufficiently large n we have $|\mathcal{P}(S; n)| = p(n)2^{n-|S|-1}$ for some polynomial $p(n)$ depending on S, but we go on to show that $p(i) = 0$ for all $i \in S$ and also that $0, 1, \ldots, i_k$ are roots of $p(n)$ for any $i_r \in S$ if $i_{r+1} - i_r$ is odd. We also probabilistically enumerate $|\mathcal{P}(S; n)|$ using alternating subsequences and explicitly compute $p(n)$ when $S = \{m, m+3, \ldots, m+3k\}$. Lastly, we prove that the coefficients of the expansion of $p(n)$ in a binomial coefficient basis are nonnegative for various S and discuss conjectures regarding the complex roots of $p(n)$.

1 Introduction

Let $\mathbb{P}$ be the positive integers and $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{P}$. Also, let $\mathfrak{S}_n$ be the *symmetric group* of all permutations $\pi = \pi_1\pi_2\cdots\pi_n$ of $[n]$. An index i of π is a *peak* if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and the *peak set* of π is defined as

$$P(\pi) = \{i : i \text{ is a peak of } \pi\}.$$

We are interested in counting the permutations of with a given peak set, so let us define

$$\mathcal{P}(S; n) = \{\pi \in \mathfrak{S}_n : P(\pi) = S\}.$$

Theorem 1.1. *If $S = \{i_1 < \cdots < i_s\}$ is admissible, then*

$$|\mathcal{P}(S; n)| = p(n)2^{n-|S|-1}$$

where $p(n) = p(S; n)$ is a polynomial depending on S such that $p(n)$ is an integer for all integral n. In addition, $\deg p(n) = i_s - 1$ (when $S = \emptyset$ we have $\deg p(n) = 0$).

It is important to note that if S is inadmissible, then $p(S; n) = 0$.

Corollary 1.2. *If $S \neq \emptyset$ is admissible and $m = \max S$, then*

$$p(S;n) = p_1(m-1)\binom{n}{m-1} - 2p_1(n) - p_2(n)$$

where $S_1 = S - \{m\}$, $S_2 = S_1 \cup \{m-1\}$, and $p_i = p(S_i, n)$ for $i = 1, 2$.

We also know that $\max S$ is a root of $p(n)$ from *Permutations with Given Peak Set*, which is important for many of the proceeding theorems.

Theorem 1.3. *If $S = \{m\}$ is admissible, then*

$$p(S;n) = \binom{n-1}{m-1} - 1.$$

We are investigating permutations with a given peak set for various reasons. An application of this permutation statistic is an insight into the randomness of medical data.

This paper has the following structure: An approach to the positivity conjecture by bounding the roots of the peak polynomial, various results regarding the roots of the peak set polynomial, various new recurrence relations for specific peak polynomials, closed form and factored polynomials for special peak sets, a probabilistic method of enumerating peak sets, and our conjectures.

2 An approach to the positivity conjecture

Billey, et. al. proposed the following conjecture in [1].

Conjecture 2.1. (Positivity Conjecture) If S is admissible, and $p(S;n)$ is expanded in the polynomial basis $\binom{n-m}{k}$ as

$$p(S;n) = \sum_{k=0}^{m-1} c_k^S \binom{n-m}{k}$$

where $m = \max S$, then all c_k^S are nonnegative integers.

The fact that all c_k^S are integers was proved in [1], leaving only the fact that the c_k^S are nonnegative to be proved.

We now state a conjecture which, if true, has the above conjecture as a corollary.

Conjecture 2.2. If S is admissible and $m = \max S$, $p(S;n)$ has no zero with real part greater than m.

We now prove that Conjecture 2.1 is a consequence of Conjecture 2.2. We will write $p(x)$ for $p(S;n)$, for convenience.

Lemma 2.3. *If a polynomial of degree $m-1$ has all of its derivatives $p'(x), p''(x), \ldots p^{(m-1)}(x)$ nonnegative for all $x > m$, then all of its finite differences $\Delta p(m), \Delta^2 p(m), \ldots \Delta^{m-1} p(m)$ are nonnegative.*

Proof. We use a specific case of Formula (44) of [2], yielding that there exists $\xi \geq m$ such that

$$k!\Delta^{k-1}p(m) = p^{(k)}(\xi)$$

Since $k!$ is nonnegative and by assumption $p^{(k)}(\xi)$ is nonnegative for $1 \leq k \leq m-1$, and $p^{(m)}(x)$ is identically zero, $\Delta^{k-1}p(m)$ must be nonnegative for all k such that $0 \leq k \leq m-1$. □

Lemma 2.4. *If none of p, p', p'', ..., $p^{(m-1)}$ has a real root greater than m, then $p(x)$, $p'(x)$, ..., $p^{(m-1)}(x)$ are all nonnegative for all $x > m$.*

Proof. Since there is always at least one permutation with an admissible peak set S, we know that $p(S;n)$ is positive for $n > m$. Since p is a nonzero polynomial, it must be either eventually positive of eventually negative. Since $p(S;n)$ is positive for arbitrarily large values of n, we know that p is eventually positive. Thus the leading coefficient of p must be positive. This implies that all of the derivatives of p are also eventually positive. If any derivative $p^{(k)}(x)$ was negative for some $x_0 > m$, then by the Intermediate Value Theorem, $p^{(k)}$ would have a zero greater than x_0, and thus greater than m, which proves the contrapositive of the statement of this lemma. □

Lemma 2.5. *If p has no zeros with real part $> m$, then p', p'', ..., $p^{(m-1)}$ have no zeros with real part $> m$, and thus no real zeros $> m$.*

Proof. We use the Gauss-Lucas theorem, which states that all the zeros of a function's derivatives all lie in the convex hull of the function's zeros in the complex plane. Thus if all the zeros of p lie in the half-plane $Re(z) \leq m$, their convex hull will also lie in the half-plane $Re(z) \leq m$, so the zeros of all the derivatives of p will lie in the half-plane $Re(z) \leq m$, so all the zeros of all the derivatives of p will have real part $\leq m$, and thus no real zeros $> m$. □

Theorem 2.6. *If S is admissible and $m = \max S$, and if $p(S;n)$ has no zero with real part greater than m, then all c_k^S are nonnegative.*

Proof. This theorem is now a straightforward consequence of Lemmas 2.3, 2.4, and 2.5. □

3 Roots of the peak set polynomials

Prove that all indices are roots. This sentence serves as a transition into the investigation of the roots of these peak polynomials. The roots are of interest for two reasons. Firstly, it appears that any index in a peak set is a root of its associated peak polynomial. Secondly, if we can prove that the greatest index in a peak set is the root with the greatest real part of the associated peak polynomial, then we can prove the positivity conjecture by analyzing the derivatives of the peak polynomial rather than its finite differences.

Also, we need make a distinction between algebraic roots and a zero from inadmissibility. For example, $p(2,5;4) = -2$, but $|\mathcal{P}(2,5;4)| = 0$ because it is not admissible.

Theorem 3.1. *If $S = \{i_1 < \cdots < i_s\}$, then all $i \in S$ are roots of $p(S;n)$.*

Proof. We will induct on i_s and use Corollary 2 and Theorem 3. In the basis case, $S = \{2\}$ because it is the smallest admissible peak set, and by Theorem 3 we know that

$$p(2; 2) = \binom{2-1}{2-1} - 1 = 0.$$

For the inductive hypothesis, assume that all $i \in S$ are roots of $p(S; n)$ for all peak sets S with maximum element $< i_s$. By Corollary 2,

$$p(S; n) = p_1(m-1)\binom{n}{m-1} - 2p_1(n) - p_2(n).$$

We know that i_s is a root of $p(S; n)$ by the extension of Theorem 1 given in *Permutations with Given Peak Set* by Billey, Burdzy, and Sagan. Let $i < i_s$ be in S. We then have

$$p(S; i) = p_1(m-1)\binom{i}{m-1} - 2p_1(i) - p_2(i).$$

By the inductive hypothesis, $p_1(i) = p_2(i) = 0$. If $i < m-1$, $\binom{i}{m-1} = 0$, and if $i = m-1$, S_1 is inadmissible, so $p_1(m-1)\binom{i}{m-1} = 0$. Thus $p(S; i) = 0$ for all $i \in S$. $\square$

Lemma 3.2. *Let* $S = \{i_1 < i_2 < \cdots < i_{s-1} < i_s\}$. *If* $i_s - i_{s-1}$ *is odd,* $0, 1, 2, \ldots i_{s-1}$ *are roots of* $p(S; n)$.

Proof. By applying the recursive formula of Corollary 1.2 repeatedly until S_2 is inadmissble, we obtain the following formula:

$$p(S; n) = -2p_1(n)\chi(i_s - i_{s-1} \text{ even}) + \sum_{i=0}^{i_s - i_{s-1} - 2} (-1)^i p_1(i_s - i - 1)\binom{n}{i_s - i - 1}$$

If $i_s - i_{s-1}$ is odd, this simplifies to

$$\begin{aligned} p(S; n) &= \sum_{i=0}^{i_s - i_{s-1} - 2} (-1)^i p_1(i_s - i - 1)\binom{n}{i_s - i - 1} \\ &= \sum_{i=0}^{i_s - i_{s-1} - 2} (-1)^i p_1(i_s - i - 1)\frac{1}{(i_s - i - 1)!}\prod_{j=0}^{i_s - i - 2}(n - j) \\ &= \prod_{j=0}^{i_{s-1} - 1}(n - j) \sum_{i=0}^{i_s - i_{s-1} - 2} (-1)^i p_1(i_s - i - 1)\frac{1}{(i_s - i - 1)!}\prod_{j=l}^{i_s - i - 2}(n - j) \end{aligned}$$

Thus $0, 1, \ldots i_{s-1} - 1$ are all roots of $p(S; n)$, and since $i_{s-1} \in S$, i_{s-1} is a root of $p(S; n)$ by Theorem 3.1, so $0, 1, \ldots i_{s-1}$ are all roots of $p(S; n)$. $\square$

Lemma 3.3. *Let $S = \{i_1 < i_2 < \cdots < i_s\}$, and let $S' = \{i_1 < i_2 < \cdots < i_s < i_{s+1}\}$. If a nonnegative integer $a \leq i_{s-1}$ is a root of $p(S; n)$, then a is a root of $p(S', n)$.*

Proof. Using the formula obtained in Lemma 3.2:

$$p(S'; n) = -2p_1(n)\chi(i_s - i_{s-1} \text{ even}) + \sum_{i=0}^{i_s - i_{s-1} - 2} (-1)^i p_1(i_s - i - 1)\binom{n}{i_s - i - 1}$$

We showed in Lemma 3.2 that the sum in this formula is zero for all nonnegative integers $n \leq i_{s-1}$. In particular, it is zero for $n = a$. Since a is a root of p_1, $-2p_1(n)\chi(i_s - i_{s-1}$ even$) = 0$, so $p(S', a) = 0$, so a is also a root of S'. $\square$

Theorem 3.4. *If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible, and $i_{j+1} - i_j$ is odd for any j such that $1 \leq j \leq s$, then $0, 1, 2, \ldots i_j$ are roots of S.*

Proof. We can now proceed by induction. Let $S^1 = \{i_1 < \cdots < i_{j+1}\}$. By Lemma 3.2, $0, 1, \ldots i_j$ are roots of $p(S^1; n)$. We now assume that $0, 1, \ldots i_j$ are roots of $p(S^k; n)$. By Lemma 3.3, $0, 1, \ldots i_j$ are thus roots of $p(S^{k+1}; n)$. So by induction $0, 1, \ldots i_j$ are roots of $p(S; n)$. $\square$

Theorem 3.5. *Let $S_L = \{i_1 < \ldots < i_l\}$, $S_R = \{j_1 = 2 < j_2 < \ldots < j_r\}$, and construct $S = \{i_1 < \ldots < i_l < j_1 + i_l + 1 < \ldots < j_r + i_l + 1\}$. Then*

$$2p(S; n) = \binom{n}{i_l + 1} p(S_L; i_l + 1) p(S_R, n - i_l - 1)$$

Proof. Select $i_l + 1$ of the n elements. There are $\binom{n}{i_l+1}$ ways to do this. Arrange them so that they have peak set S_L. There are $P(S_L; i_l + 1)$ ways to do this. Arrange the remaining $n - (i_l + 1)$ elements so that they have peak set S_R. There are $P(S_R; n - i_l - 1)$ ways to do this. Now concatenate the two permutations to form a permutation on n elements which has peak set S. This construction yields all permutations with peak set S, and since i_l and $j_1 + i_l + 1$ are peaks, $i_l + 1$ and $j_1 + i_l$ are not, so this construction only gives permutations with peak set S. Thus we have

$$P(S; n) = \binom{n}{i_l + 1} P(S_L; i_l + 1) P(S_R, n - i_l - 1)$$

In terms of the peak polynomials, this is

$$2^{n-s-1} p(S; n) = \binom{n}{i_l + 1} 2^{i_l + 1 - l - 1} p(S_L; i_l + 1) 2^{n - i_l - 1 - r - 1} p(S_R; n - i_l - 1)$$

Since $s = |S| = l + r$, this simplifies to

$$2p(S; n) = \binom{n}{i_l + 1} p(S_L; i_l + 1) p(S_R, n - i_l - 1)$$

$\square$

Corollary 3.6. *If $S = \{i_1 < \ldots < i_s\}$, and there are two consecutive indices i_j and i_{j+1} such that $i_{j+1} - i_j = 3$, and if we define $S + 1 = \{i + 1 : i \in S\}$, we have*

$$p(S + 1; n) = Cnp(S; n - 1)$$

where $C = C(S)$ does not depend on n.

Proof. If we let $i_l = i_j$ and $j_k = i_{j+k}$ for $1 \leq k \leq s - j$, we can divide S into S_L and S_R, and divide $S + 1$ into $S_L + 1$ and S_R. We then apply the theorem to obtain:

$$p(S + 1; n) = \frac{1}{2}\binom{n}{i_l + 2} p(S_L + 1; i_l + 2) p(S_R; n - i_l - 2)$$

whereas

$$\begin{aligned} p(S; n - 1) &= \frac{1}{2}\binom{n-1}{i_l + 1} p(S_L; i_l + 1) p(S_R; n - i_l - 2) \\ \Rightarrow p(S + 1; n) &= \frac{1}{i_l + 2} \frac{p(S_L + 1; i_l + 2)}{p(S_L; i_l + 1)} np(S; n - 1) \\ &= Cnp(S; n - 1) \end{aligned}$$

where $C = \frac{1}{i_l+2} \frac{p(S_L+1; i_l+2)}{p(S_L; i_l+1)}$ depends only on S_L, not on n. □

Corollary 3.7. *Let $S_L = \{i_1 < \ldots < i_l\}$ and $S_R = \{j_1 = 2 < \ldots j_r\}$. If S_R has no roots with real part greater than j_r, then if we construct $S = \{i + 1 < \ldots < i_l < j_1 + (i_l + 3 - j_1) < j_2 + (i_l + 3 - j_1) < \ldots < j_b + (i_l + 3 - j_1)\}$, $p(S; n)$ has no real roots with real part greater than $max(S) = j_r + i_l + 3 - j_1 = j_r + i_l + 1$.*

Proof. $\binom{n}{i_l+1}$ has zeroes at $0, 1, \ldots i_l$ and $p(S_L; i_l + 1) > 0$, so
$p(S; n)$ will have a zero with real part greater than $j_r + i_l + 1$
$\Leftrightarrow p(S_R; n - i_l - 1)$ has a zero with real part greater than $j_r + i_l + 1$
$\Leftrightarrow p(S_R; n)$ has a zero with real part greater than j_r. □

4 Additional recursions

While studying the roots of peak set polynomials, we have constructed new recurrence relations for specific peak sets, e.g. ending in an odd gap. Most of the recurrences factor out integer roots with the rest of the equation defined by some recurrence. We also gain some insight on how the peak polynomial grows from n to $n + 1$ by looking at the ratio of consecutive terms, which eliminates the unknown coefficient.

Theorem 4.1. *If S is admissible and $n > \max S$, then*

$$2p(S; n + 1) = (n + 1)p(S; n) - p(S \cup \{n\}; n + 1).$$

Proof. Select n of the $n+1$ elements and arrange them such that they have peak set S. Add the leftover element to the end of this sequence. The resulting sequence will have peak set either S or $S \cup \{n\}$, and any sequence with peak set S or $S \cup \{n\}$ can be constructed this way. There are $\binom{n+1}{n} = n+1$ ways to choose n of the $n+1$ elements, and $|\mathcal{P}(S;n)|$ ways to arrange them so that they have peak set S, so we have

$$|\mathcal{P}(S;n+1)| + |\mathcal{P}(S \cup \{n\};n+1)| = (n+1)\,|\mathcal{P}(S;n)|\,.$$

Using Theorem 1.1, we see that

$$2^{(n+1)-s-1}p(S;n+1) + 2^{(n+1)-(s+1)-1}p(S \cup \{n\};n+1) = (n+1)2^{n-s-1}p(S;n),$$

where $s = |S|$. Dividing by 2^{n-s-1} yields

$$2p(S;n+1) + p(S \cup \{n\};n+1) = (n+1)p(S;n),$$

so

$$2p(S;n+1) = (n+1)p(S;n) - p(S \cup \{n\};n+1).$$

□

It should be noted that this formula can also be obtained by plugging $p(S \cup \{n+1\};n+1)$ into Corollary 1.2 and noting that $S \cup \{n+1\}$ is inadmissible, but this proof is still necessary because the proof of that formula assumed we were using an admissible set.

Corollary 4.2. *If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible, then*

$$p(S;i_s+2) = \frac{i_s+2}{2}p(S;i_s+1).$$

Proof. This follows directly from Theorem 4.1 by plugging i_s+1 in for n and observing that $S \cup \{i_s+1\}$ is inadmissible. □

Corollary 4.3. *If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible, then*

$$|\mathcal{P}(S;i_s+2)| = (i_s+2)\,|\mathcal{P}(S;i_s+1)|\,.$$

Proof. This is a direct consequence of Theorem 1.1 and Corollary 4.2. □

From a combinatorial perspective, Corollary 4.3 makes complete sense because i_s is a peak, which implies that i_s+1 is not peak. Furthermore, i_s+2 cannot be a peak because we are counting permutations in $\mathfrak{S}_{i_s+2}$. There are i_s+2 ways to choose i_s+1 symbols from i_s+2, and there are $|\mathcal{P}(S;i_s+1)|$ ways to arrange them into S. The final symbol is appended to the permutation and will never be a peak.

Now we will look at $p(S;n)$ when S has a certain kind of final gap. More specifically, we are interested in peak sets whose roots are all integral, which seems to only happen when $S = \{i_1 < i_2 < \cdots < i_s < i_s+3\}$ and $S = \{i_1 < i_2 < \cdots < i_s < i_s+3 < i_s+5\}$.

Lemma 4.4. *If $S = \{i_1 < \cdots < i_s < i_s + 3\}$ is admissible, then*

$$p(S; n) = p_1(i_s + 2)\binom{n}{i_s + 2} - p_1(i_s + 1)\binom{n}{i_s + 1}.$$

Proof. We use Corollary 1.2 twice and observe that

$$\begin{aligned}
p(S; n) &= p_1(i_s + 2)\binom{n}{i_s + 2} - 2p_1(n) - p_2(n) \\
&= p_1(i_s + 2)\binom{n}{i_s + 2} - 2p_1(n) - \left(p_1(i_s + 1)\binom{n}{i_s + 1} - 2p_1(n) - 0\right) \\
&= p_1(i_s + 2)\binom{n}{i_s + 2} - p_1(i_s + 1)\binom{n}{i_s + 1}.
\end{aligned}$$

Notice that the rightmost polynomial terminates because $\{i_1 < \cdots < i_s < i_s + 1\}$ is inadmissible. □

Theorem 4.5. *If $S = \{i_1 < \cdots < i_s < i_s + k\}$ for some odd $k \geq 3$, then*

$$p(S; n) = \frac{\prod_{i=0}^{i_s}(n - i)}{(i_s + 1)!} \sum_{i=1}^{k-1} (-1)^k p_1(i_s + i) \frac{\prod_{j=1}^{i-1}(n - (i_s + j))}{\prod_{j=2}^{i}(i_s + j)}.$$

Proof. We will induct on k and use Corollary 1.2 and Lemma 4.4. In the basis case $k = 3$, and by Lemma 4.4 we see that

$$\begin{aligned}
p(S; n) &= p_1(i_s + 2)\binom{n}{i_s + 2} - p_1(i_s + 1)\binom{n}{i_s + 1} \\
&= \frac{\prod_{i=0}^{i_s}(n - i)}{(i_s + 1)!}\left[\frac{p_1(i_s + 2)(n - (i_s + 1))}{i_s + 2} - p_1(i_s + 1)\right].
\end{aligned}$$

In the inductive step we use the recurrence formula to produce a term known by the inductive hypothesis.

$$\begin{aligned}
p(S; n) &= p_1(i_s + k - 1)\binom{n}{i_s + k - 1} - p_1(i_s + k - 2)\binom{n}{i_s + k - 2} + p(S_1 \cup \{i_s + k - 2\}; n) \\
&= p_1(i_s + k - 1)\binom{n}{i_s + k - 1} - p_1(i_s + k - 2)\binom{n}{i_s + k - 2} \\
&\quad + \frac{\prod_{i=0}^{i_s}(n - i)}{(i_s + 1)!} \sum_{i=1}^{k-3} (-1)^k p_1(i_s + i) \frac{\prod_{j=1}^{i-1}(n - (i_s + j))}{\prod_{j=2}^{i}(i_s + j)} \\
&= \frac{\prod_{i=0}^{i_s}(n - i)}{(i_s + 1)!} \sum_{i=1}^{k-1} (-1)^k p_1(i_s + i) \frac{\prod_{j=1}^{i-1}(n - (i_s + j))}{\prod_{j=2}^{i}(i_s + j)}.
\end{aligned}$$

□

The following two corollaries illustrate the factorization of $p(S;n)$ when $S=\{i_1<i_2<\cdots<i_s<i_s+3\}$ or $S=\{i_1<i_2<\cdots<i_s<i_s+3<i_s+5\}$. All of the roots of these polynomials are nonnegative integers, and the only the coefficient is defined recursively.

Corollary 4.6. *If $S=\{i_1<\cdots<i_s<i_s+3\}$, then*

$$p(S;n)=\frac{p_1(i_s+1)}{2(i_s+1)!}(n-(i_s+3))\prod_{i=0}^{i_s}(n-i).$$

Proof. We see that

$$p(S;n)=\frac{\prod_{i=0}^{i_s}(n-i)}{(i_s+1)!}\left[\frac{p_1(i_s+2)(n-(i_s+1))}{i_s+2}-p_1(i_s+1)\right]$$

using Theorem 4.5, but we also know that i_s is a root of $p(S;n)$. It follows that

$$\begin{aligned}p(S;n)&=\frac{\prod_{i=0}^{i_s}(n-i)}{(i_s+1)!}\left[\frac{p_1(i_s+2)(n-(i_s+1))-p_1(i_s+1)(i_s+2)}{i_s+2}\right]\\&=\frac{\prod_{i=0}^{i_s}(n-i)}{(i_s+1)!}\left[\frac{p_1(i_s+2)}{i_s+2}\left(n-\left(i_s+1+\frac{p_1(i_s+1)(i_s+2)}{p_1(i_s+2)}\right)\right)\right].\end{aligned}$$

Because there is only one remaining root,

$$i_s+3=i_s+1+\frac{p_1(i_s+1)(i_s+2)}{p_1(i_s+2)},$$

so it follows that

$$\begin{aligned}p(S;n)&=\frac{p_1(i_s+2)}{(i_s+2)!}(n-(i_s+3))\prod_{i=0}^{i_s}(n-i)\\&=\frac{p_1(i_s+1)}{2(i_s+1)!}(n-(i_s+3))\prod_{i=0}^{i_s}(n-i)\end{aligned}$$

by Corollary 4.2. □

It should be noted that $i_s+3=i_s+1+\frac{p_1(i_s+1)(i_s+2)}{p_1(i_s+2)}$ also implies Corollary 4.2. Recall that $p_1(n)=p(\{i_1<\cdots<i_s\};n)$. So for any $S=\{i_1<\cdots<i_s\}$, we can simply insert i_s+3 into S and use the previous equation. Also, $n=i_s+1$ is the first value where $p(S;n)$ is nontrivial and makes combinatorial sense.

Corollary 4.7. *If $S=\{i_1<\cdots<i_s<i_s+3<i_s+5\}$, then*

$$p(S;n)=\frac{p(S-\{i_s+3,i_s+5\};i_s+1)}{3(i_s+1)!}(n-(i_s+3))(n-(i_s+5))(n-(i_s-2))\prod_{i=0}^{i_s}(n-i).$$

Proof. This is a direct consequence of Theorem 1.2 and Corollary 4.6. □

The following equations define the the peak polynomials recursively if their peak sets are of the appropriate form.

Corollary 4.8. *If $S = \{i_1 < \cdots < i_s < i_s + 3\}$ is admissible, then*

$$p(S; n+1) = \frac{(n+1)(n-(i_s-2))}{(n-i_s)(n-(i_s+3))} p(S; n).$$

Proof. The constant coefficients and intermediate terms cancel in $p(s; n+1)/p(s; n)$ using Corollary 4.6. □

Corollary 4.9. *If $S = \{i_1 < \cdots < i_s < i_s + 3 < i_s + 5\}$ is admissible, then*

$$p(S; n+1) = \frac{(n+1)(n-(i_s+2))(n-(i_s+4))(n-(i_s-3))}{(n-i_s)(n-(i_s+3))(n-(i_s+5))(n-(i_s-2))} p(S; n).$$

Proof. The constant coefficients and intermediate terms cancel in $p(s; n+1)/p(s; n)$ using Corollary 4.7. □

5 Polynomials for specific peak sets

Theorem 5.1. *If $S = \{m, m+3, \ldots, m+3k\}$ with $k \geq 1$, then*

$$p(S; n) = \frac{(m-1)(n-(m+3k))}{2(m+1)!(12^{k-1})} \prod_{i=0}^{m+3(k-1)} (n-i).$$

Proof. We induct on k and use Theorem 1.3 and Lemma 4.4. If $k = 1$, then $S = \{m, m+3\}$, and we know from Theorem 1.3 that

$$\begin{aligned}
p(m, m+3; n) &= p_1(m+2)\binom{n}{m+2} - p_1(m+1)\binom{n}{m+1} \\
&= \left[\binom{m+1}{m-1} - 1\right]\binom{n}{m+2} - \left[\binom{m}{m-1} - 1\right]\binom{n}{m+1} \\
&= \left[\frac{(m-1)(m+2)}{2}\right]\binom{n}{m+2} - \left[\frac{2(m-1)}{2}\right]\binom{n}{m+1} \\
&= \frac{(m-1)(n-(m+3))}{2(m+1)!} \prod_{i=0}^{m} (n-i).
\end{aligned}$$

In the inductive step we know $p_1(n)$ from the inductive hypothesis because $S_1 = \{m, m+3, \ldots, m+3(k-1)\}$. Using Lemma 4.4, it follows that

$$\begin{aligned} p(S;n) &= p_1(m+3k-1)\binom{n}{m+3k-1} - p_1(m+3k-2)\binom{n}{m+3k-2} \\ &= \frac{m-1}{2(m+1)!(12^{k-2})}\left[\frac{2(n-(m+3k-2))}{4!} - \frac{1}{3!}\right]\prod_{i=0}^{m+3(k-1)}(n-i) \\ &= \frac{(m-1)(n-(m+3k))}{2(m+1)!(12^{k-1})}\prod_{i=0}^{m+3(k-1)}(n-i), \end{aligned}$$

which is the desired result. $\square$

Theorem 5.2. *If $S = \{m, m+3, \ldots, m+3k, m+3k+2\}$ with $k \geq 1$, then*

$$p(S;n) = \frac{(m-1)(n-(m+3k-5))(n-(m+3k))(n-(m+3k+2))}{(m+1)!(12^k)}\prod_{i=0}^{m+3(k-1)}(n-i).$$

Proof. Using Corollary 1.2, we see that

$$p(S;n) = p_1(m+3k+1)\binom{n}{m+3k+1} - 2p_1(n)$$

because $p_2(n)$ is inadmissible. We know $p_1(n)$ from Theorem 5.1, so it follows that

$$\begin{aligned} p(S;n) &= \frac{(m-1)(n-(m+3k))}{2(m+1)!(12^{k-1})}\prod_{i=0}^{m+3(k-1)}(n-i)\left[\frac{1}{3!}(n-(3k-1))(n-(3k-2))-2\right] \\ &= \frac{(m-1)(n-(m+3k-5))(n-(m+3k))(n-(m+3k+2))}{(m+1)!(12^k)}\prod_{i=0}^{m+3(k-1)}(n-i). \end{aligned}$$

$\square$

6 Probabilistic enumeration

If $S = \{i_1 < i_2 < \cdots < i_s\}$ is an admissible peak set, then let $\mathcal{Q}(S;n)$ be the set of permutations $\pi \in \mathfrak{S}_n$ whose peak set contains S. In symbols,

$$\mathcal{Q}(S;n) = \{\pi \in S_n : S \subseteq P(\pi)\}.$$

Now let us partition S into runs of alternating subsequences. An *alternating subsequence* of a peak set is subset of S such that

$$A_r = \{i_r, i_r+2, \ldots, i_r+2(k-1)\}$$

where $i_r - i_{r-1} > 2$ if $i_{r-1} \in S$, and we call A_r an *alternating subsequence* because

$$\pi_{i_r-1} < \pi_{i_r} > \pi_{i_r+1} < \pi_{i_r+2} > \cdots < \pi_{i_r+2(k-1)} > \pi_{i_r+2(k-1)+1}$$

is an *alternating permutation* in $\mathfrak{S}_{2k+1}$. Let $\mathcal{A}(S)$ be the function that partitions S into alternating subsequences. For example, if $S = \{2, 4, 7, 9, 11, 15\}$, then $\mathcal{A}(S) = \{A_1, A_3, A_6\} = \{\{2, 4\}, \{7, 9, 11\}, \{15\}\}$.

Theorem 6.1. *If S is admissible, then*

$$|\mathcal{Q}(S; n)| = n! \prod_{A_r \in \mathcal{A}(S)} \frac{T_{2|A_r|+1}}{(2|A_r| + 1)!}$$

where T_k is a tangent number.

Proof. Let $S = \{i_1 < i_2 < \cdots < i_s\}$, $A_r = \{i_r, i_r + 2, \ldots, i_r + 2(k-1)\}$ be the alternating subsequence in $\mathcal{A}(S)$ containing i_r, and π be a random permutation in $\mathfrak{S}_n$. Clearly $A_r \subseteq P(\pi)$ if and only if $\pi_{i_r-1} < \pi_{i_r} > \pi_{i_r+1} < \pi_{i_r+2} > \cdots < \pi_{i_r+2(k-1)} > \pi_{i_r+2(k-1)+1}$, so the probability that $A_r \subseteq P(\pi)$ depends only on these $2k + 1$ values of π. There are T_{2k+1} alternating permutations in $\mathfrak{S}_{2k+1}$, so the probability that $2k+1$ elements form an alternating permutation is $T_{2k+1}/(2k + 1)!$.

Now we will look at a different alternating subsequence in $\mathcal{A}(S)$, so let $A_{r'} = \{i_{r'}, i_{r'} + 2, \ldots, i_{r'} + 2(k' - 1)\}$. By an earlier argument, we know the probability that $A_{r'} \subseteq P(\pi)$ depends only on the consecutive $2k'+1$ different values of π, and we know that this probability is $T_{2k'+1}/(2k' + 1)!$. Moreover, we see that these probabilities are independent because $\{\pi_{i_r-1}, \pi_{i_r}, \ldots, \pi_{i_r+2(k-1)+1}\} \cap \{\pi_{i_{r'}-1}, \pi_{i_{r'}}, \ldots, \pi_{i_{r'}+2(k'-1)+1}\} = \emptyset$, which we know from the definition of alternating subsequences.

It follows that the probability that $S \subseteq P(\pi)$ is equal to the product of the individual probabilities that $P(\pi)$ contains the different alternating subsequences because they are independent, and so we see that $|\mathcal{Q}(S; n)|$ is the expected value of the probability that $S \subseteq P(\pi)$, which completes the proof. $\square$

It's worth noting that if an alternating subsequence has k peaks, then the probability that its associated permutation in $\mathfrak{S}_{2k+1}$ is an alternating permutation is the k^{th} zero-based coefficient of the Maclaurin series of $\tan(x)$. Consequently, we use tangent numbers. For example, the probability that a random permutation has peaks at i and $i+2$ is $2/15$ (assuming that it is admissible), and we see that

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 \cdots .$$

Corollary 6.2. *If S is admissible, then*

$$|\mathcal{P}(S; n)| = \sum_{T \supseteq S} (-1)^{|T-S|} |\mathcal{Q}(T; n)|.$$

Proof. This is a direct consequence of Theorem 6.1 and the Principle of Inclusion and Exclusion. □

Corollary 6.3. *If $S = \{i_1 < \cdots < i_s\}$ is admissible and $\sum_{A_r \in \mathcal{A}(S)} 2|A_r| + 1 = i_s + 1$, then*

$$|\mathcal{P}(S; i_s + 1)| = (i_s + 1)! \prod_{A_r \in \mathcal{A}(S)} \frac{T_{2|A_r|+1}}{(2|A_r| + 1)!}.$$

Proof. We use Corollary 6.2 above and observe that S is the only admissible superset of S. □

Discuss why the gap of three is so special.

7 Conjectures

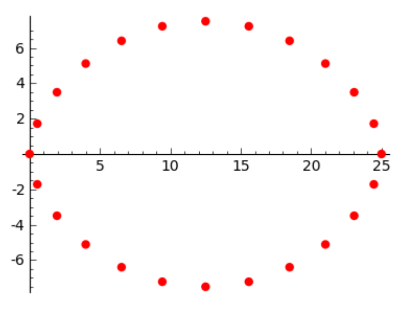
Conjecture 7.1. If $S = \{2, 5, \ldots, 2 + 3k\}$ for some $k \geq 0$, then S is the most frequent peak set in $\mathfrak{S}_{3(k+1)}$.

We know that the number permutations with this peak set is $(3(k+1))!/3^{k-1}$ by Corollary 6.3.

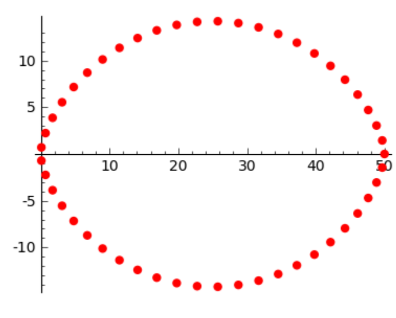
Conjecture 7.2. If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible, then the real part of every root of $p(S; n)$ is less than or equal to i_s.

Conjecture 7.3. If $S = \{i_1 < i_2 < \cdots < i_s\}$ is admissible and all of the roots of $p(S; n)$ are real, then all of the roots of $p(S; n)$ are integral. Furthermore, this only occurs when $S = \{2\}$, $S = \{2, 4\}$, $S = \{3\}$, $S = \{3, 5\}$, $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3\}$, or $S = \{i_1 < i_2 < \cdots < i_s < i_s + 3 < i_s + 5\}$.

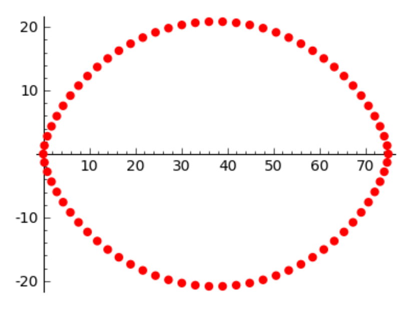
Conjecture 7.4. If $S = \{m\}$, then the roots of $p(S; n)$ lay on the boundary of a football-shaped curve in the complex plane. Furthermore, 0 and m are the only real roots if m is odd, and m is the only real root if m is even.



$S = \{25\}$



$S = \{50\}$



$S = \{75\}$

From Theorem 1.3 we know that if $S = \{m\}$, then $p(S;n) = \binom{n-1}{m-1} - 1$. When looking at the complex roots of this polynomial, it makes sense to let $n = z$, where z is a complex variable. It follows that the values of z that we are seeking satisfy

$$\frac{\Gamma(z)}{\Gamma(m)\Gamma(z-m+1)} - 1 = 0,$$

and furthermore

$$\Gamma(z) = \Gamma(z-m+1)m!.$$

Conjecture 7.5. *Only complex roots may fall in the final gap?

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